

Vector Basics

The purpose of this handout is to provide a review of vectors and basic vector operations. This review assumes that you have previously learned vectors in a mathematics class. Those aspects of vectors that are important to meteorology are reviewed here.

I. Definitions

A. Scalars

If a quantity is completely determined by its magnitude, it is a *scalar*. In meteorology quantities such as temperature, pressure, mixing ratio, and many other quantities are scalars.

B. Vectors

If a quantity is completely determined by both magnitude and direction, it is a *vector*. In meteorology quantities such as velocity, acceleration, and force are vectors.

Vectors are usually represented by a bold face capital letter. In this review the letter \mathbf{V} will be used to represent any generic vector. Lower case letters will be used to represent the magnitude of the vector (v) or its scalar components.

II. Basic Operations

A. Graphical versus Mathematical Representations

Vectors can be presented in at least two ways:

1) A vector can be displayed graphically as an arrow where the orientation of the arrow represents direction and the length of the arrow is the magnitude. [Figure 1]

Direction is measured relative to some reference. For wind velocity this reference is usually north. A graphical vector is plotted from some point (o in Figure 1) which is assumed to be the location of the vector quantity.

2) Mathematically a vector is often represented as a set of points that corresponds to the components of the vector. See the next section for a discussion of components.

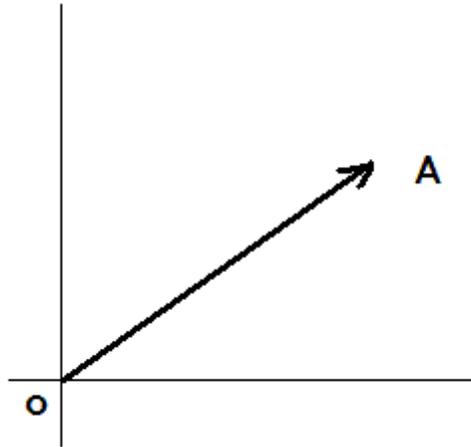


Figure 1: Graphical Vector

For example, the vector $\mathbf{V} = (v_1, v_2, v_3)$ is a mathematical representation of the vector \mathbf{V} with three components: v_1 , v_2 and v_3 . Some texts stack the components vertically, making it look more like a matrix:

$$\mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

B. Components

Vectors are usually presented relative to some coordinate system. Assume that we are working in a standard Cartesian coordinate system with x increasing toward the right and y increasing toward the top. [Figure 2]

To determine the components of a vector, you set the origin of the coordinate system at the origin of the vector and drop perpendiculars to each axis as shown in Figure 2 for a two-dimensional vector. You can then draw a new vector from the origin to the point where the perpendicular crosses the x axis and another vector from the origin to the point where the perpendicular crosses the y axis. These two vectors are the component vectors of the original vector relative to the specified coordinate system.

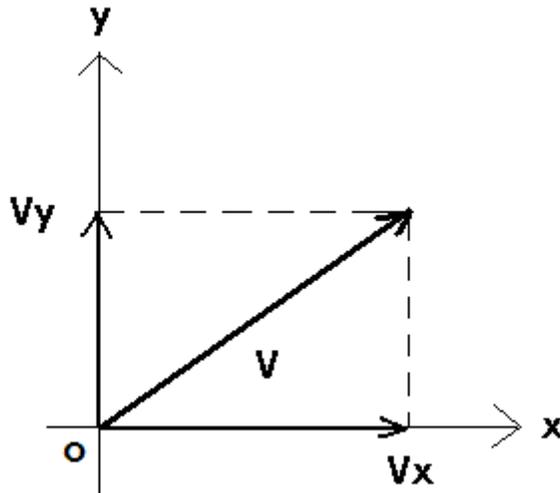


Figure 2: Vector Components

Determining components graphically is fairly easy. Given the vector and a coordinate system, just drop perpendiculars to the axes from the point of the vector and draw the component vectors. To mathematically determine the magnitude of the component vectors, you need an additional piece of information. Specifically, you need the angle of the vector relative to one of the axes. You can then use trigonometry to calculate the components.

The x value where the perpendicular crosses the x axis is the magnitude of the x -component vector, v_x , and the y value where the perpendicular crosses the y axis is the magnitude of the y -component vector, v_y . In mathematical format, $\mathbf{V} = (v_x, v_y)$. These components, v_x and v_y , are the scalar components of \mathbf{V} .

Once you have the magnitude of the component vectors, you can represent a vector in a third way, using base vectors. The simplest set of base vectors is the unit vector. Unit vectors are orthogonal and normalized. They are vectors along each axis with a magnitude of 1 unit of measurement (whatever that measurement unit is). You can then represent a three dimensional vector as $\mathbf{V} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$.

C. *Wind Velocity*

In meteorology it is often easier to present wind fields as velocity vectors. We will reserve the term *wind velocity* for the vector wind and the term *wind speed* for the magnitude of the wind vector. Wind velocity is actually a three-dimensional

quantity, that is two horizontal components and one vertical component, but in most situations we may limit our use of the term to two dimensions.

On a weather map we represent horizontal wind using a wind barb. The wind barb shows direction by its orientation and speed by the number of lines or flags on the bar. This representation looks like the tail of an arrow. [See Figure 3a]

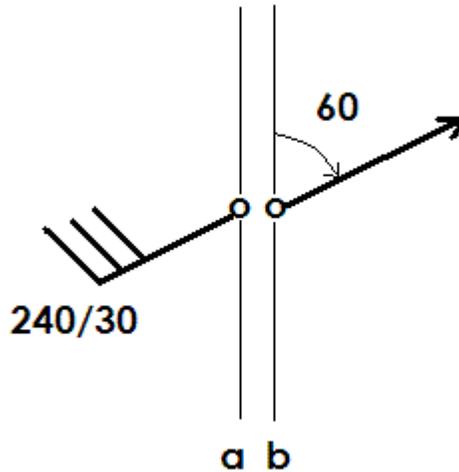


Figure 3: Wind Vectors

For operational wind data, you must be careful when you determine wind components. For example, a 240° wind blows from the west-southwest but its vector representation is toward 60° or east-northeast. [See Figure 3b] Thus, if you want to determine the east-west and north-south components, you first have to construct a true vector from the operational wind information and then calculate the appropriate components.

D. Vector Addition and Subtraction

Vectors may be added (or subtracted) in two ways: graphically or mathematically. A graphical presentation is useful when you are discussing vector topics, as in the classroom. However, if you need exact values for a vector, the mathematical approach is required.

Figure 4 shows the addition of two vectors, \mathbf{V}_1 and \mathbf{V}_2 . To add two vectors, first plot one of the vectors. Then plot the second vector with its origin at the tip of the first vector. A line connecting the origin of the first vector to the tip of the second vector is the sum of these two vectors.

Figure 4 also shows the components of the vectors V_1 and V_2 using dashed lines to indicate the x and y magnitudes. Note that when you sum up the x components of the two individual vectors, the sum equals the x component of the vector sum. Similarly, the sum of the y components of the individual vectors is also the y component of the vector sum. Expressed mathematically:

$$V_{xsum} = V_{x1} + V_{x2} \quad V_{ysum} = V_{y1} + V_{y2}$$

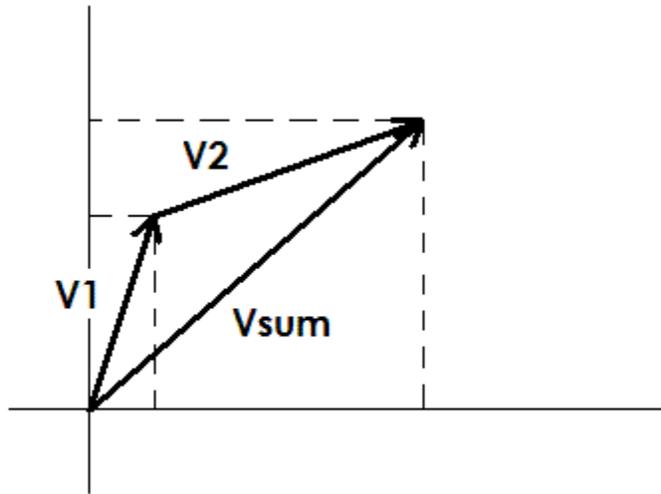


Figure 4: Addition of Vectors

Subtraction of two vectors is similar to addition. Consider the following vector relationship:

$$V_{dif} = V_1 - V_2 = V_1 + (-V_2)$$

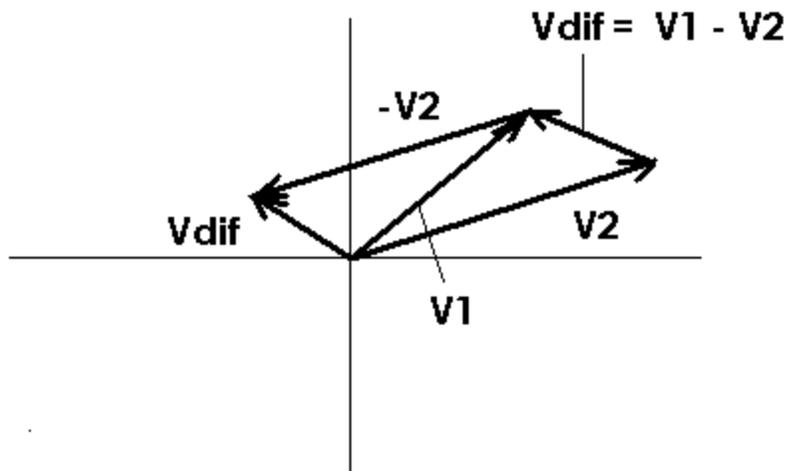


Figure 5: Subtraction of Vectors

Figure 5 illustrates this vector relationship. It adds the negative of \mathbf{V}_2 to \mathbf{V}_1 . The difference between these vectors can also be drawn as a line from the tip one vector to the tip of the other vector. This line is the vector difference between the two original vectors. The direction of this vector is from \mathbf{V}_2 to \mathbf{V}_1 . An easy way to remember the direction of the difference vector is to recall that:

$$\mathbf{V}_1 = \mathbf{V}_2 + \mathbf{V}_{\text{dif}}$$

E. Dot Products

The dot (or scalar) product of two vectors is represented by:

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = v_1 v_2 \cos(e) = v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \mathbf{V}_2 \cdot \mathbf{V}_1$$

where e is the angle between the two vectors, \mathbf{V}_1 and \mathbf{V}_2 , and x , y and z are the components of each vector in three-dimensional space. A dot product of two vectors results in a scalar.

From a graphical perspective, the dot product is the product of the magnitude of one vector times the component of the other vector along the first vector. [See Figure 6]

If the two vectors are perpendicular, the dot product is zero.

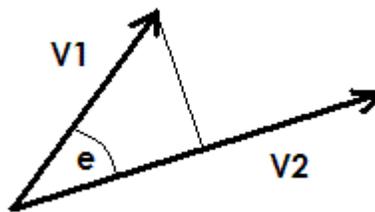


Figure 6: Vector Cross Product

F. Cross Products

The cross (or vector) product of two vectors is represented by:

$$\mathbf{V}_1 \times \mathbf{V}_2 = v_1 v_2 \sin(e) \mathbf{l} = - \mathbf{V}_2 \times \mathbf{V}_1$$

where e is the angle between the two vectors, \mathbf{V}_1 and \mathbf{V}_2 . The unit vector \mathbf{l} is perpendicular to the plane of \mathbf{V}_1 and \mathbf{V}_2 and is in the direction determined by the right-hand rule. A cross product of two vectors results in a vector.

If the two vectors are parallel, their cross product is zero.

III. Partial Derivatives

A partial derivative is directional derivative. It is usually represented by the symbol ∂ . Many of the equations used in meteorology use partial derivatives.

When taking a partial derivative of some function with respect to one variable, hold the other variables constant. Let's look at an example to illustrate these operations:

Assume that $f(x,y,z) = x^2y + y^3 \sin(z^2)$. For first partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 + 3y^2 \sin(z^2)$$

$$\frac{\partial f}{\partial z} = 2y^3z \cos(z^2)$$

If you continue this process to the second derivative level:

$$\frac{\partial^2 f}{\partial x^2} = 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x$$

$$\frac{\partial^2 f}{\partial y^2} = 6y \sin(z^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x$$

IV. Differentiation

A. The Basics

Vectors can be differentiated similar to any mathematical function. Some basic rules include:

$$\frac{d}{dt} (\mathbf{V}_1 + \mathbf{V}_2 + \dots) = \frac{d\mathbf{V}_1}{dt} + \frac{d\mathbf{V}_2}{dt} + \dots$$

$$\frac{d}{dt} (\mathbf{V}_1 \cdot \mathbf{V}_2) = \left(\frac{d\mathbf{V}_1}{dt} \cdot \mathbf{V}_2 \right) + \left(\mathbf{V}_1 \cdot \frac{d\mathbf{V}_2}{dt} \right)$$

$$\frac{d}{dt} (\mathbf{V}_1 \times \mathbf{V}_2) = \left(\frac{d\mathbf{V}_1}{dt} \times \mathbf{V}_2 \right) + \left(\mathbf{V}_1 \times \frac{d\mathbf{V}_2}{dt} \right)$$

B. The Del Operator

$$\mathbf{DEL} = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

C. Operations

1. Gradient Operator

$$\mathbf{GRAD} = \mathbf{DEL}(\text{scalar}) = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k}$$

The gradient of a scalar quantity is a vector directed from low toward high values of the scalar, perpendicular to isopleths of the scalar field (in a horizontal plane).

Temperature and pressure gradients are measured using **GRAD**. The mathematical symbol for **GRAD** is an inverted Δ (like an unfilled ∇).

2. Divergence

$$\mathbf{DIV} = \mathbf{DEL} \cdot \mathbf{V} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

The divergence of a vector is a scalar quantity. In meteorology divergence is a kinematic property of the wind field.

3. Curl

$$\begin{aligned} \mathbf{CURL} &= \mathbf{DEL} \times \mathbf{V} \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k} \end{aligned}$$

The curl of a vector is a vector quantity. In meteorology curl is a kinematic property of the wind field called vorticity.

4. Laplacian

$$\text{Laplacian} = \mathbf{DEL} \cdot \mathbf{DEL}(\text{scalar})$$

$$= \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}$$

The Laplacian is a scalar and is the divergence of the gradient vector.

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